Comput. Methods Appl. Mech. Engrg. 199 (2010) 1091–1099

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

A gPC-based approach to uncertain transonic aerodynamics

F. Simon [a], P. Guillen [b], P. Sagaut [a], D. Lucor [a,*]

[a] Institut Jean Le Rond d'Alembert, UMR 7190, Université Pierre et Marie Curie, Paris 6, F-75005 Paris, France
[b] ONERA, Applied Aerodynamics Department, F-92322 Châtillon, France

ARTICLE INFO

Article history:
Received 27 October 2008
Received in revised form 17 September 2009
Accepted 25 November 2009
Available online 11 January 2010

Keywords:
Transonic airfoil aerodynamics
Uncertain quantification
Stochastic collocation
Polynomial Chaos

ABSTRACT

The present paper focus on the stochastic response of a two-dimensional transonic airfoil to parametric uncertainties. Both the freestream Mach number and the angle of attack are considered as random parameters and the generalized Polynomial Chaos (gPC) theory is coupled with standard deterministic numerical simulations through a spectral collocation projection methodology. The results allow for a better understanding of the flow sensitivity to such uncertainties and underline the coupling process between the stochastic parameters. Two kinds of non-linearities are critical with respect to the skin-friction uncertainties: on one hand, the leeward shock movement characteristic of the supercritical profile and on the other hand, the boundary-layer separation on the aft part of the airfoil downstream the shock. The sensitivity analysis, thanks to the Sobol' decomposition, shows that a strong non-linear coupling exists between the uncertain parameters. Comparisons with the one-dimensional cases demonstrate that the multi-dimensional parametric study is required to get the correct shape and magnitude of the standard deviation distributions of the flow quantities such as pressure and skin-friction.

1. Introduction

Uncertainty quantification (UQ) of the influence of uncertain parameters onto physical systems is a major issue in order to properly predict the system response to random inputs. Several benefits can be obtained from such studies. For example, it allows to introduce realistic safety margins depending on the solution sensitivity to the random inputs. One of the main interests is also to study the coupling process between several uncertain parameters which can not be investigated through linearized approaches like Adjoint-state-based methods. Moreover, UQ leads to a classification of the most influential parameters on the system response and allows for the identification of extreme behaviors under specific coupling. Moreover, UQ leads to a classification of the most influential parameters on the system response.

Stochastic aerodynamics, i.e. the study of aerodynamic properties of an immersed solid body in the presence of uncertainties, is a recent field compared with classical deterministic aerodynamics. While deterministic aerodynamics is mostly concerned with an almost exact prediction of the flow, stochastic aerodynamics aims at predicting the most probable flow features, the mean flow features (averaging being performed over the different values taken by the uncertain parameters) but also extreme events. Most related works were carried out in the incompressible flow framework for wind engineering oriented studies, e.g. see Solari and Piccardo [21] and Pagnigni and Solari [18] and references given therein. In these works, the main purpose is to model turbulent wind gusts that are responsible for structural loads and to predict the induced structure deformations and/or vibrations. An important point is that in these works the aerodynamic forces exerted on the solid body are obtained via easy-to-solve surrogate models, but not computed using a Navier–Stokes solver. This approach is relevant when the emphasis is put on integrated parameters such as drag and lift for bluff bodies, for which accurate response surfaces can be built.

The present paper addresses another issue of stochastic aerodynamics which is of great interest for aerospace engineering related studies, i.e. the prediction of a transonic flow around a 2D clean wing in the presence of external flow related uncertainties. The emphasis is put on the features of the Reynolds-averaged mean flow, which are the useful and meaningful data used for aerodynamic analysis and shape optimization. The problem of wind gust buffeting is not considered here. For such an analysis, global or oversimplified surrogate models are no longer relevant, and high fidelity Navier–Stokes simulations must be carried out, since engineers are interested in getting a detailed prediction of the flow structure for shape optimization purposes.

Walters and Huyse [24] have underlined the necessity of uncertainty quantification in computational fluid dynamics (CFD) and have reviewed the few methods available. Among them, the Polynomial Chaos (PC) theory introduced by Wiener [26] has been applied to fluid mechanics problems. It has appeared well suited to

* Corresponding author. Tel.: +33 1 44 27 87 12.
E-mail addresses: franck.simon@cea.fr (F. Simon), philippe.guillen@onera.fr (P. Guillen), pierre.sagaut@upmc.fr (P. Sagaut), didier.lucor@upmc.fr (D. Lucor).

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doi:10.1016/j.cma.2009.11.021

get insight into the influence of random variables on relevant aerodynamic issues.

The PC approach is very well suited for the representation of gaussian processes. Its extension to non-gaussian random uncertainties by Xiu and Karniadakis [29] is called generalized Polynomial Chaos (gPC). A development of the whole basic principles can be found in the book by Ghanem and Spanos [5] and an outlook of uncertainty quantification in CFD through the use of (gPC) representation can be found in Knio and Le Maître [8].

This methodology has been successfully applied in the last decade to a wide range of fluid mechanics problems, allowing researchers to assess the sensitivity of a system to random/uncertain external conditions as well as the influence of numerical parameters.

Le Maître et al. [10] have investigated the transport and mixing process in a microchannel whereas oscillations of random amplitude have been used by Wan and Karniadakis [25] to study the stochastic heat transfer enhancement in a grooved channel. Lucor and Karniadakis [14] have investigated the influence of a uncertain freestream velocity on the vortex shedding process behind a circular cylinder while Le Maître et al. [11] have investigated the Rayleigh-Bénard instability with random wall temperature. More recently, Ko et al. [9] have performed simulations of a 2D mixing layer with random boundary conditions to access the shear layer growth sensitivity to stochastic inflow forcing. Moreover, a different approach has been used by Lucor et al. [16] who have studied the sensitivity of a LES solution to subgrid-scale-model parametric uncertainty. They have quantified the influence of an uncertain Smagorinsky constant value on the energy spectra in the case of isotropic homogeneous decaying turbulence. In this work, the authors attempt to measure the sensitivity of the system response not to external conditions/inputs uncertainty but to the model uncertainty itself. This is not the point of the present paper. We emphasize that our goal is not to access the quality of our deterministic solver for a particular or several configurations. Instead, we wish to quantify the robustness of the simulated response to external parametric uncertainty.

Uncertainty quantification is essentially a study of errors, both their description and their consequences. It can be viewed as the determination of error bars to be assigned to the numerical solution algorithms. This problem is particularly difficult for non-linear hyperbolically dominated flows, Yu et al. [30]. Very few papers deal with stochastic compressible flows. Mathelin et al. have applied the Galerkin PC representation to quasi-one-dimensional supersonic flow, Mathelin et al. [17]. For this problem, they have also derived a collocation technique that reduces the computational burden associated with high-order non-linearities. Some research work in the supersonic regime have also been performed by Lin et al. [12] dealing with 2D Euler equations for a stochastic wedge flow (random inflow velocity and random oscillations of the wedge around its apex). Loeven et al. [13] make use of a deterministic compressible RANS code which is coupled to a probabilistic collocation solver to propagate freestream aerodynamic (Mach number) uncertainty through a subsonic steady flow around a NACA0012 airfoil. The Mach number takes a uniform distribution form with a 5% coefficient of variation and the angle of attack is deterministic, $\alpha = 5$ deg. The stochastic solution converges fast and exhibits no spatial discontinuity. For compressible flows with shocks, such as transonic flows, global gPC approximation can suffer from lack of robustness due to stochastic oscillating systems involving long-term integration and/or discontinuities in the random space. Some work has been pursued to tackle this issue by designing adaptive stochastic method that can handle discontinuities. For stochastic collocation techniques, we can mention among others the work of Foo et al. [4], Witteveen and Bijl [27]. Poëtte et al. [19] propose a stochastic intrusive approach to tackle shocks in compressible gas dynamics. Their gPC-based technique relies on the decomposition of the entropic variable of the flow and does not depend on a special discretization of the random space.

For the optimization of stochastic compressible flows, it is crucial to ensure that the optimal model response is robust with respect to the inherent uncertainties associated with the design variables, constraints and the objective function. Traditional optimization techniques together with UQ are computationally expensive and time consuming when it comes to identify what drives the response variability. A stochastic optimization framework combining stochastic surrogate model representation and optimization algorithm is proposed by Lucor et al. [15]. A gPC stochastic representation is used as the surrogate model. This approach allows both sensitivity and optimization analysis. The stochastic optimization method is applied to a multi-layer reacting flow device. The geometric configuration is assumed to be uncertain and the structure design is optimized to maximize the energy transfer between the reacting flow and the device moving parts.

The aim of the present study is to quantify the response of the flow around a classical bi-dimensional airfoil to uncertain flow conditions in the transonic regime with the use of a stochastic collocation spectral projection based on the gPC theory. The new difficult technical problem addressed in this article is therefore to assess the capability of a pseudo-spectral method like gPC to accurately capture the non-linear stochastic behavior of flows with strong discontinuities like shocks, the shock being very sensitive to uncertainties. This sensitivity result in dramatic changes in both shock location and shock intensity, making the gPC convergence process much more complex that for non-bifurcating smooth flows. The first part of the article is devoted to the simulations overview. The main part of the study focus on the physical analysis of the simulations for a 2-parameter stochastic case where both the infinite Mach number M_∞ and the angle of attack α are assumed to be stochastic parameters with uniform distribution. Then, conclusions are drawn.

2. Numerical procedure

2.1. Simulations overview

The bi-dimensional airfoil retained to perform the current study is the supercritical OAT15A profile with a chord $c = 0.23$ m. The freestream conditions are the same as those previously used for wind tunnels experiments (Jacquin et al. [6]) as well as numerous numerical simulations with $P_i = 1$ bar and $T_i = 300$ K. The Mach number M_∞ and the angle of attack α are, respectively equal to 0.73 and 2.5 degrees. The Reynolds number Re_c based on c is equal to 3×10^6. In the following, the deterministic simulation with $(M_\infty, \alpha) = (0.73, 2.5)$ will be referred to as the reference simulation. A realistic range of variation for the uncertain parameters will be chosen as to make sure that buffeting does not occur within the parametric region.

Since the emphasis is put on the Reynolds-Averaged flow features, Reynolds-Averaged Navier–Stokes (RANS) are retained as the relevant mathematical model in this work. Let us also emphasize that stochastic convergence is expected for RANS solution, but would not for instantaneous turbulent fields due to the chaotic nature of the latter. The compressible RANS equations are solved using the ElsA aerodynamic solver developed at ONERA for the past ten years, Cambier and Veuillot [1]. A Jameson spatial scheme is used along with the one equation Spalart–Allmaras model. The 2D mesh is composed of two blocks the size of which are, respectively 385×161 cells (C block surrounding the airfoil) and 129×369 cells in the wake, leading to 110,000 cells and has been extended to 80 chords in all directions. The mesh and the accuracy of the CFD tool has been carefully checked in past studies, Deck [2].

The capability of the turbulence model, the numerical scheme and computational grid to accurately predict the RANS steady solution at each quadrature point of the gPC projection (see below) has been assessed. Moreover, we have checked that the stochastic predictions of our study are not too sensitive to our RANS parameters as long as we use a sufficiently resolved spatial discretization.

2.2. Polynomial Chaos representation and collocation projection

The following section presents a brief description of the mathematical framework of stochastic spectral method employing expansions of the random inputs and solution based on Askey-type orthogonal polynomial functionals of random vectors. The (gPC) theory is a generalization of the Hermite Chaos originally proposed by Wiener [26] and the reader should refer to Ghanem and Spanos [5] for more details. The stochastic collocation spectral method essentially transform the stochastic problem to a high-dimensional deterministic problem through the use of appropriate projections. The dimensionality of the new system is a function of the noise level of the random input and the order of accuracy required from the representation.

We consider a probability space $(\Omega, \mathscr{A}, \mathscr{P})$ where Ω denotes the event space, $\mathscr{A} \subset 2^{\Omega}$ its σ-algebra and $\mathscr{P}$ its probability measure. Let $p(\omega)$ be a random field, i.e. mappings $p : \Omega \rightarrow V$ from the probability space into a function space V. If $V = \mathbb{R}$, $p(\omega)$ are random variables, and if V is a function space over a time and/or a space interval, random fields are stochastic processes. V is a Hilbert space with dual V', norm $\|\cdot\|$ and inner product $(\cdot,\cdot) : V \times V \rightarrow \mathbb{R}$. As V is densely embedded in V', we abuse notation and denote by $(\cdot,\cdot)$ also the $V \times V'$ duality pairing.

In the present study, we will consider second-order random fields, i.e. $p : \Omega \rightarrow V$ is a second-order random field over V, if

$$\mathbb{E}\|p\|^2 = \mathbb{E}(p,p) < \infty, \tag{1}$$

where $\mathbb{E}$ denotes the expectation of a random variable $Y \in L^1(\Omega, \mathbb{R})$ and is defined by

$$\mathbb{E}Y = \int_{\omega\in\Omega} Y(\omega) d\mathscr{P}(\omega) = \int Y(\xi) W(\xi) d\xi, \tag{2}$$

with $W(\xi)$ is the measure of the random variable ξ denoting the density of the law $\mathscr{P}(\omega)$ with respect to the Lebesgue measure $d\xi$ and with integration taken over a suitable domain, determined by the range of ξ.

The (gPC) representation is a useful means of representing second-order random fields $p(\omega)$ parametrically through the set of random variables $\{\xi_j(\omega)\}_{j=1}^{\infty}$, through the events $\omega \in \Omega$. We have:

$$p(\omega) = \sum_{j=0}^{\infty} p_j \phi_j(\boldsymbol{\xi}(\omega)). \tag{3}$$

Here $\{\phi_j(\xi(\omega))\}$ are mutually orthogonal polynomials satisfying the orthogonality relation:

$$\mathbb{E}\phi_i\phi_j = \delta_{ij}\mathbb{E}\phi_i^2. \tag{4}$$

Practically, the expansion in Eq. (3) is then truncated to a finite-dimensional space based on a "finite-dimensional noise assumption": i.e. only a finite number N of random variables $\{\xi_j(\omega)\}_{j=1}^{\infty}$ are used. Further, the highest polynomial order P is selected based on accuracy requirements. Consequently, if we denote by $p(\vec{\mathbf{x}})$, the spatial pressure field, the finite-term *gPC* expansion reads as follows:

$$p(\vec{\mathbf{x}}, \xi) = \sum_{j=0}^{M-1} p_j(\vec{\mathbf{x}}) \Phi_j(\xi(\omega)), \quad \text{with} \quad M = \frac{(N+P)!}{N!P!}. \tag{5}$$

The probabilistic collocation method was first introduced by Tatang et al. [23]. It consists in constructing polynomial approximations of the solution from a nodal set of collocation points. After collocation projections, the resulting set of deterministic equations is always uncoupled and each solution is obtained with a deterministic numerical solver. The continuous solution is then interpolated on the data points using multi-dimensional tensor product Lagrange basis. Evaluation of the solution moments requires integrating those Lagrange basis, which can be quite cumbersome, unless we choose the nodal set of points to be a cubature set points. By choosing the cubature weight function to coincide with the joint density of the random input, the computation of the moments becomes straightforward. Nevertheless, the interpolation error is hard to control with this approach, Xiu and Hesthaven [28].

In this study, we use the gPC-collocation method which is a pseudo-spectral method with gPC polynomial basis. It is somewhat similar to the previous approach as it relies on evaluating the solution at finite number of quadrature points but is not based on Lagrange interpolation. Instead, we construct the expansion (5) based on the solver's evaluations during a post-processing stage. In the case where both M_∞ and α are random variables ($N = 2$), the coefficients p_j of (5) can be directly expressed as:

$$\forall j \in \{0,\ldots,M-1\}, \quad p_j(\vec{\mathbf{x}}) = \frac{\langle p(\vec{\mathbf{x}},\omega)\Phi(\mathbf{M}_\infty(\omega),\boldsymbol{\alpha}(\omega))\rangle}{\langle \Phi_j^2(\mathbf{M}_\infty(\omega),\boldsymbol{\alpha}(\omega))\rangle} \tag{6}$$

This method is *non-intrusive* in the sense that we project the stochastic solution directly onto each member of the orthogonal basis chosen to span the random space. It has the advantage not to require modifications to the existing deterministic solver. The global error of the final representation can be seen as a superposition of an aliasing error (coming from the interpolation), a finite-term projection error (due to the truncated representation) and a numerical error due to the intrinsic numerical approximation of the deterministic solver.

Different ways of dealing with high-dimensional integrations can be considered depending on the prevalence of accuracy vs. efficiency, Keese [7]. Here, we use of a full numerical Gauss quadrature due to its efficiency for moderate N values. The number of quadrature points N_q, representing the number of simulations and relying on the regularity of the function to integrate, is fixed by the user.

In the present study, statistical moments up to the second-order have been investigated thanks to the following expression:

$$\mu_{p(\vec{\mathbf{x}})} = \langle p(\vec{\mathbf{x}},\omega)\rangle = p_0(\vec{\mathbf{x}}), \tag{7}$$

$$\sigma^2_{p(\vec{\mathbf{x}})} = \langle (p(\vec{\mathbf{x}},\omega) - p_0(\vec{\mathbf{x}}))^2\rangle = \sum_{j=1}^{M-1}\left[p_j^2(\vec{\mathbf{x}})\langle\Phi_j^2\rangle\right]. \tag{8}$$

Probability density functions (pdf) of the solutions can easily be evaluated as well from (5).

2.3. Simulations under uncertainties

In this study, we consider the case of a 2D foil in a randomly perturbed flow in transonic regime. We suppose that the stochastic perturbation affects the magnitude and direction of the incoming inflow velocity. Such random fluctuations can be characteristic of some gust of wind. This translates to a random component in both the flow speed (or Mach number) and the angle of incidence of the flow (or angle of attack of the profile). In aeroelasticity, turbulent wind gusts can be modeled by time-dependent forcings represented by random processes, Soong and Grigoriu [22]. In this case, the emphasis is put on the structural response to the stochastic excitation. In these studies, the stochastic model can be quite elaborate but the flow around the airfoil is not directly computed. This approximation may prove disastrous in the case of a transonic

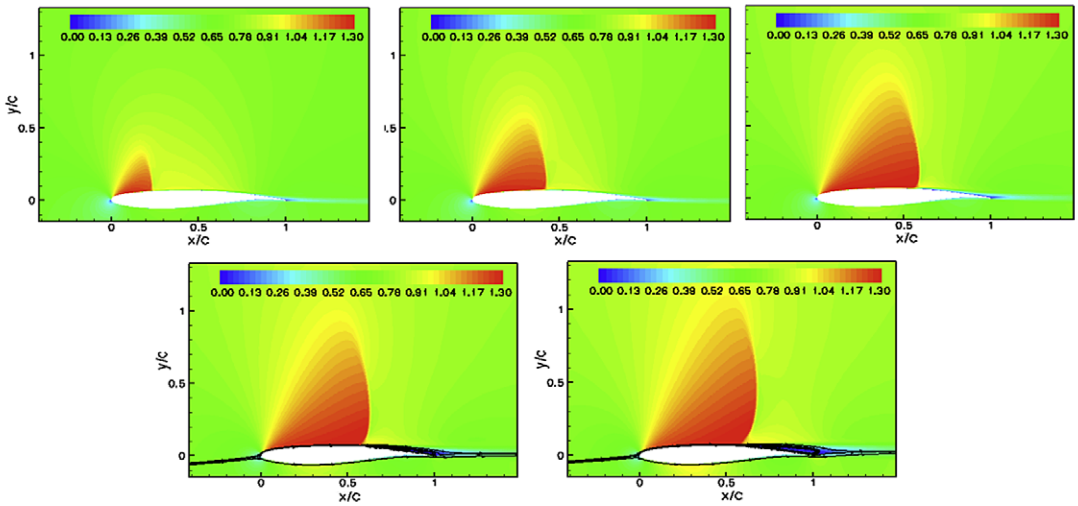

Fig. 1. Mach number isocontour fields for 5 different inflow Mach number conditions. From left to right and top to bottom: $M_\infty = 0.73(1-8\%)$; 0.73(1–4%); 0.73; 0.73(1 + 4%); 0.73(1 + 8%).

regime that is highly sensitive to perturbations implying shock-waves and separations.

Our approach is different, in the sense that we consider simpler models for the random parameters but we use a Navier–Stokes solver to propagate these uncertainties to the flow around the foil. We treat the uncertain input parameters to our simulations as i.i.d random variables. The non-linearity of the system then transforms these uncorrelated random variables to spatial random processes.

The range of variation of the uncertain parameters is chosen as to avoid the buffeting region for which the RANS solver would not be accurate. This requirement suggests considering bounded supports. Once the bounds of the intervals are chosen we are left with the choice of relevant distributions. While different distributions for different parameters are not incompatible with the gPC formulation, we prefer to consider uniform distributions for both parameters. This choice means that we do not favor any particular parametric value within the domain of interest. Moreover, the choice of a uniform distribution is justified as it is the maximum entropy distribution for any continuous random variable on an interval of compact support. In other words, an assumption of any other prior distribution satisfying the constraints will have a smaller entropy, thus containing more information and less uncertainty than the uniform distribution, Jaynes [3]. The uncertain parameters retained for the current study are M_∞ and α. The Mach number M_∞ has a 0.73 mean value and a ±5% variability and the angle of attack α has a 2.5 degrees mean value and a ±20% variability. Due to the choice of uniform distributions for the inputs and without any *a priori* knowledge of the outputs pdf solution, an appropriate basis from a mathematical point of view is the Legendre polynomial basis Xiu and Karniadakis [29]. This latter one will be used in the following as our expansion basis. Additional studies were also completed for mono-dimensional cases (i.e. the uncertainties on M_∞ and α have been studied separately) and will be used for comparison in the sensitivity analysis in the last section.

In order to highlight the stochastic analysis that will follow, a short description of the flow physics is useful. Fig. 1 shows the Mach number isocontour fields for five different inflow Mach numbers M_∞ ranging from 0.73(1–8%) to 0.73(1 + 8%) with an increment of 4% and a common incidence of $\alpha = 2.5$ degrees. In all cases a supersonic area (in red) is present on the leeward side. This supersonic region ends with a shock downwards. As the Mach number increases, the supersonic region widens with the terminal shock moving downwards to a limit value. Once this limit value has been reached (for $M_\infty \approx 0.73$), a separated area (in blue) appears along the foil to the right of the shock and expands as the inflow Mach number increases. By opposition, the windward side evolves very little and does not exhibit any non-linear features.

3. Stochastic response to uncertainties

3.1. Numerical convergence

A full convergence study of the stochastic problem has been performed through the analysis of the aerodynamics coefficients as well as their first and second-order statistical moments. The two relevant parameters are N_q and P. Due to the dependence of the standard deviation accuracy onto the number of terms $(M-1)$ and so indirectly on the order P, it is of primary importance to check the influence of P on the magnitude of σ.

In this work, N_q and P have been determined sequentially in a two-step process. First, the minimum acceptable value for N_q is identified so that converged mean values of pressure and skin-friction coefficients on the foil surface are obtained. Figs. 2 and 3, respectively plot the values for these two coefficients vs. the chord abscissa for an uncertain Mach number with uniform distribution within the range 0.73 ± 8%. Converged solutions are easily obtained (with a very few number of Gauss points) along most of the profile except in the region of the shock movement for the pressure and

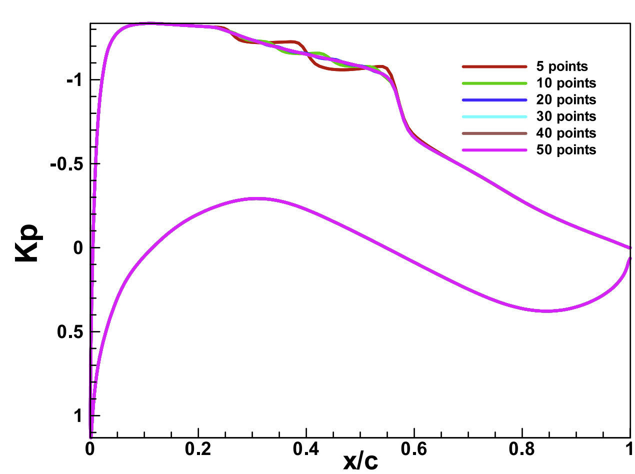

Fig. 2. Mean value distribution of the pressure coefficient along the chord.

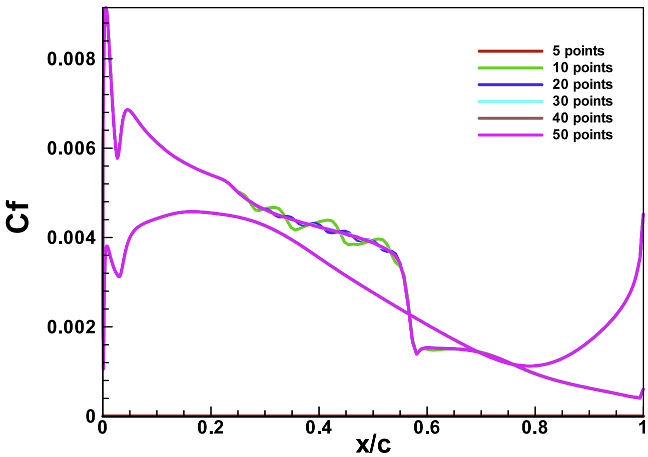

Fig. 3. Mean value distribution of the skin-friction along the chord.

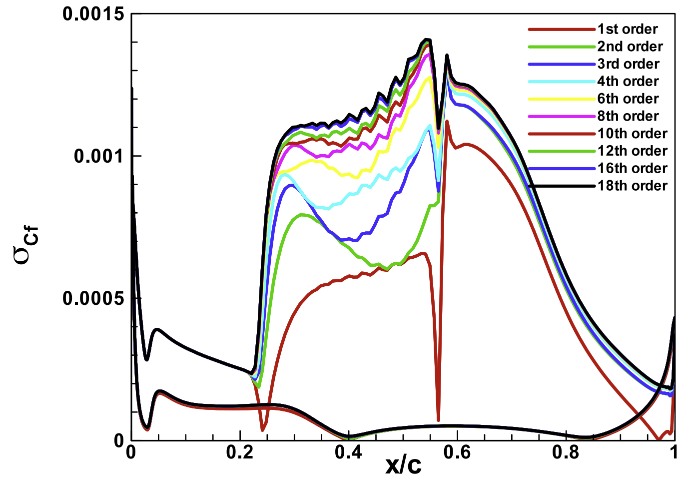

Fig. 5. Standard deviation of the skin-friction along the chord.

also in the separated area zone for the skin-friction coefficient. In these two critical regions an N_q value of 40 is required. Using this value of $N_q = 40$, the standard deviations obtained along the chord for the pressure and the skin-friction are plotted for different values of the gPC polynomial order P in Figs. 4 and 5. Similarly to the previous step, fast convergence is reached in linear regions, i.e. regions without shock and/or separation. In the critical non-linear regions an 18th polynomial order may be required.

Some results from the literature suggest that global pseudo-spectral gPC approximation based on collocation is not appropriate in the case of discontinuous or sharp solutions. In this case, the continuous interpolated solution (5) may exhibit some oscillations inducing irregular and unphysical patterns in the spatial distribution of the solution moments or pdf, e.g. stair-like profile for the mean solution (cf. Figs. 2 and 3 at low N_q). The location of these irregularities coincide with the collocation points, see for instance Witteveen and Bijl [27]. The phenomenon is particularly noticeable for local physical quantity (such as Cf), more sensitive to discretization errors.

In the following paragraph, we refer to the simplified diagram of Fig. 6 for visual assistance. Given a fixed spatial discretization grid of typical resolution size Δx along the chord, the accuracy of the gPC approximation depends on P and N_q. Let us call $p_{s_q} \equiv p(x_{s_q}, Z_q)$ the value of the discontinuous solution at the location of the shock x_{s_q} obtained for the collocation point Z_q. When the number N_q of collocation points is not sufficient and the response of the system is very sensitive to the uncertainty, it may happen that the typical distance between two neighboring shocks Δx_{s_q} is much larger than Δx. In this case, $\Delta x_{s_q} \gg \Delta x$ and the problem described hereinbefore appears, Poëtte et al. [19]. However, several of our studies (not all presented here) have shown that the profiles recover regularity when we increase N_q as $\Delta x_{s_q} \rightarrow \Delta x$. This is the case here (cf. Figs. 2 and 3 at high N_q). For some higher moments (cf. Fig. 5) and some pdf contours (cf. Fig. 8), some small oscillations may remain along the distribution, but the right profile magnitude is captured for sufficiently high P. In the case where $\Delta x_{s_q} \ll \Delta x$, one faces aliasing error as the shocks are not assigned to the correct cell.

It is in general difficult to predict the appropriate N_q as the average Δx_{s_q} is not known a priori. This latter depends on the distribution of the chosen quadrature rule as well as the sensitivity of the response to the parametric uncertainty. This sensitivity relates to the span length of the geometric envelop in which all probable discontinuous events may take place. Non-linearity of the system, monotonicity of the solution with respect to the parametric variation and airfoil geometry will affect differently this range.

In conclusion, it is somewhat possible to alleviates the problem but there exists a strong coupling between the discretization in physical space and the collocation grid in random space. As a re-

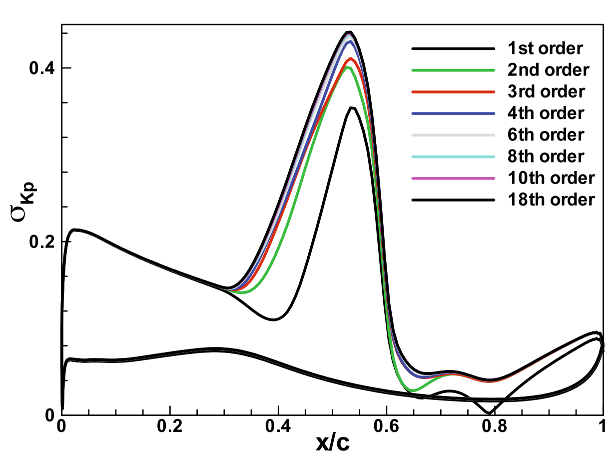

Fig. 4. Standard deviation of the pressure coefficient along the chord.

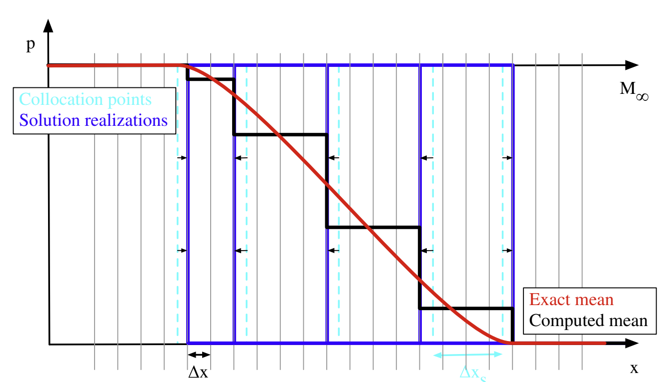

Fig. 6. Schematic illustrating the dependence between physical (Δx) and stochastic (Δx_{s_q}) discretizations. When $\Delta x_{s_q} \gg \Delta x$, i.e. the parametric (here M_∞) collocation points distribution (dotted cyan curves) is such that the corresponding discontinuous realizations are far apart (blue curves), the distribution of the gPC solution moments may exhibit some irregular and unphysical patterns (black curve). These irregularities are smoothed out (red curve) if $\Delta x_{s_q} \approx \Delta x$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

sult, refinement of the grid in one of the two spaces must happen together within the other one. In our case, due to the sensitivity of the OAT15A profile, Δx has to be small enough to properly capture shocks at the right location within the wide spatial range of variability. Therefore, it requires a fine stochastic collocation grid as was seen in the results presented in this section. This approach is still manageable for problems with few random dimensions but would become impractical and too costly for higher dimensional problems.

3.2. Mean fields, standard deviations and PDF distributions

Figs. 7 and 8 present the mean and standard deviation distribution of wall data along the airfoil for the stochastic bi-dimensional case as well as their associated PDFs distributions and PDFs profiles for five locations on the leeward side. Reference cases are also included.

Let's focus on the wall pressure coefficient *Kp* first. For $0.35 < x/c < 0.65$, we notice that stochastic solutions greatly depart from the reference solution. We also notice that the uncertain mean solution differs from the deterministic one. The main discrepancy consists in a less pronounced compression region surrounding the mean shock position. This result is consistent with the fact that the shock location is not fixed for different low Mach number realisations. Indeed, the shock moves upstream as the Mach number value decreases from the averaged value of 0.73 to its lower bound. No clear influence of the parametric uncertainty can be observed for $x/c > 0.65$ as well as for $x/c < 0.35$ which is the most upstream shock position for the uncertainty range investigated here. These observations have to be related to the behavior of the standard deviations. For $0 < x/c < 0.35$, σ_{Kp} results from a linear response of the flow to the uncertainty as the shock-wave never penetrates this area whatever M_∞ values in our range. Downstream this location, higher magnitudes of the standard deviation are observed. It appears that the strong spatial non-linearities introduced by the shocks translate to the random domain. The dependency of the shock location to M_∞ accounts for the high sensitivity of the flow in this wide region. Further downstream, σ_{Kp} becomes very weak except for $x/c > 0.9$ where trailing-edge effects can be observed.

The skin-friction behavior is quite different and exhibits discrepancies with the reference solution for both upstream and downstream locations compared to the deterministic shock position. For $x < 0.65$, these differences can be explained in a similar way as the ones for the pressure coefficient. The uncertainty range has a diffusion-like effect on the skin-friction gradient in the shock region thus explaining the lower C_f values ahead of $x/c = 0.55$. For $x/c > 0.65$, the stochastic solution presents higher C_f magnitude than the deterministic case meaning that the uncertain parameters deeply influence the boundary-layer state in the second half of the airfoil. The analysis of Mach fields (not shown here) for several realizations within the uncertainty range shows that the shock position has the tendency to shift downstream when M_∞ is raised up from its lower bound to its mean value of 0.73. For higher M_∞ values, its position remains stationary and only the boundary-layer state behind the shock is altered, leading to separation. Such evolution can be clearly evidenced when looking at the Mach contours of the mono-dimensional cases. It shows why the skin-friction

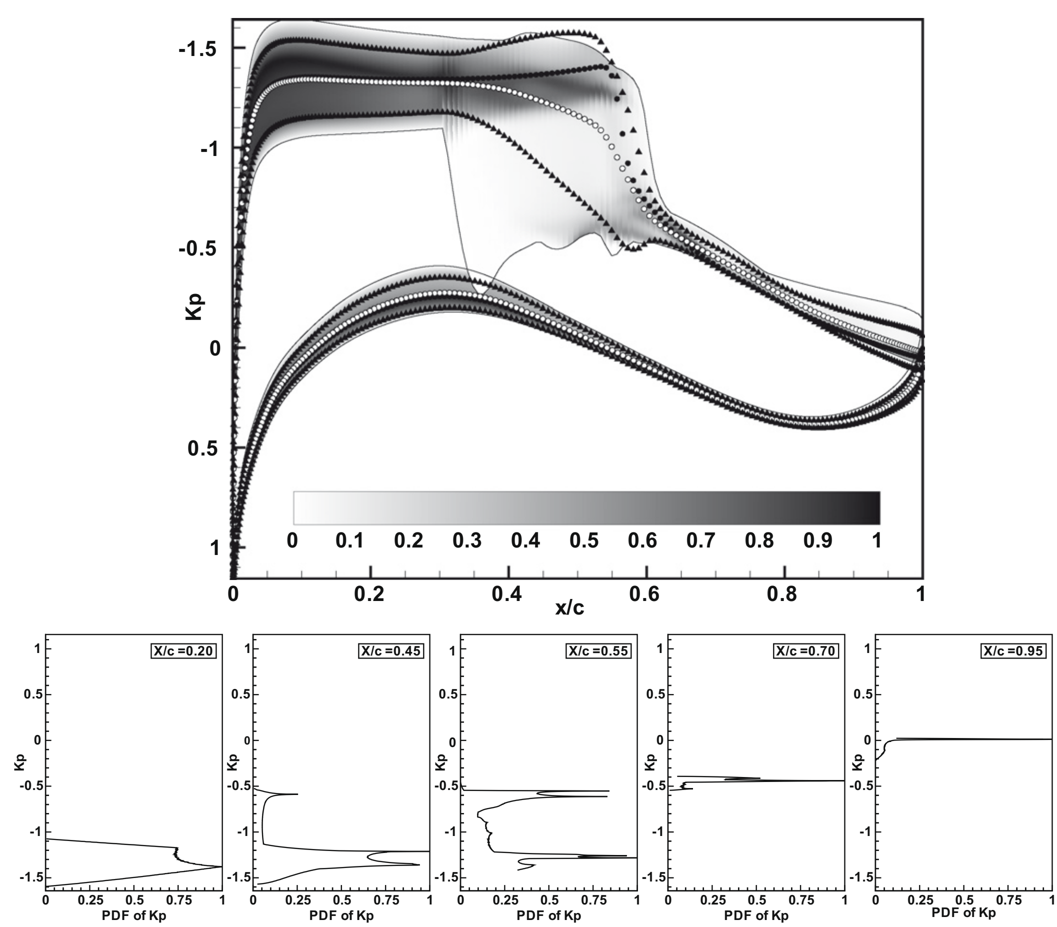

Fig. 7. The global normalized PDF contours of K_p due to uncertain M_∞ and α and the corresponding local PDF profiles at five chord locations (○: stochastic mean – ▲: stochastic mean ± standard deviation – •: reference deterministic case – —-: PDFs minimum and maximum values).

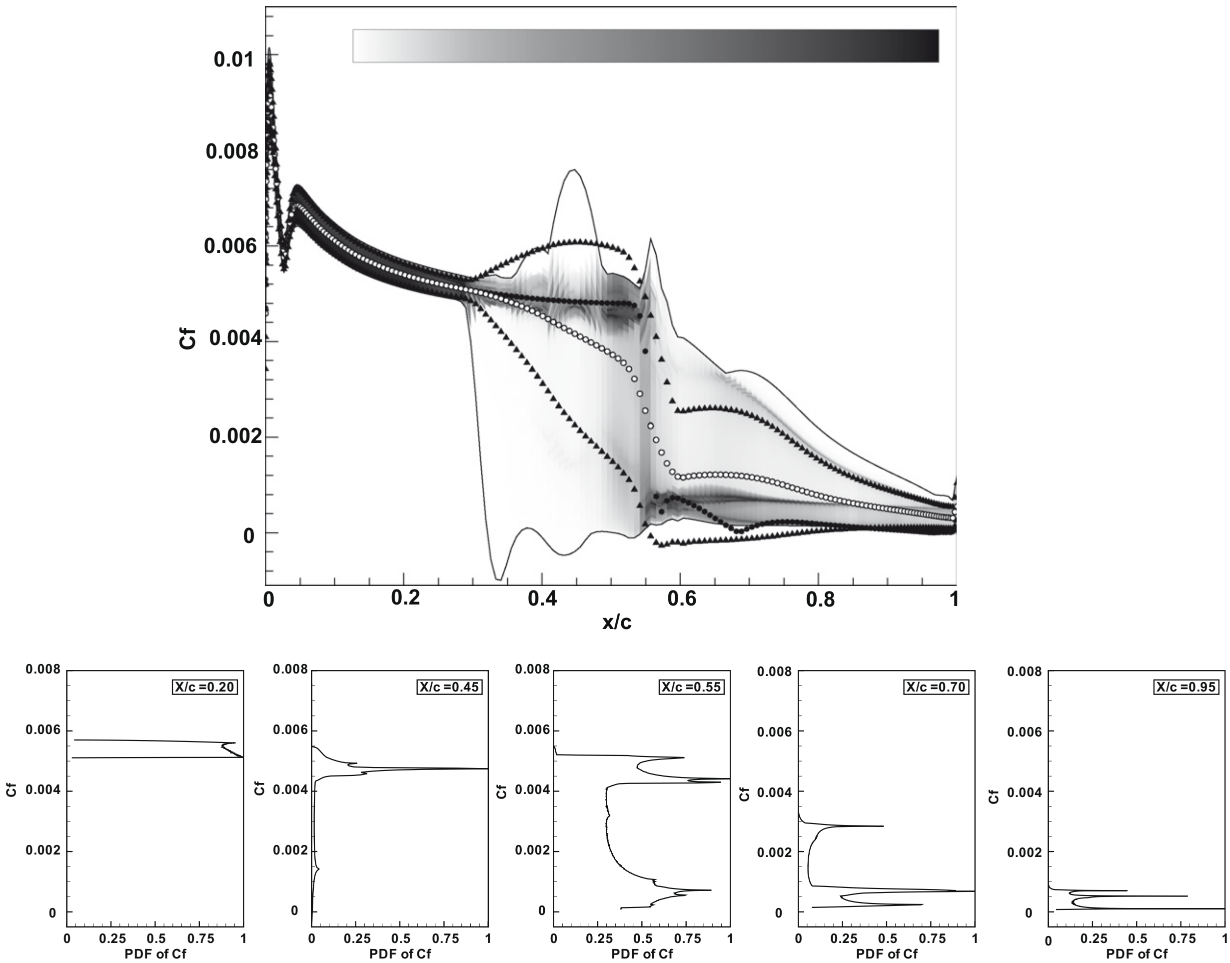

Fig. 8. The global normalized PDF contours of C_f due to uncertain M_∞ and α and the corresponding local PDF profiles at five chord locations (○: stochastic mean – ▲: stochastic mean ± standard deviation – •: reference deterministic case – —-: PDFs minimum and maximum values).

coefficient is much more sensitive to the uncertainty than the pressure coefficient in the second half of the profile. This trend is similar for the standard deviation distributions where high magnitudes of σ_{Cf} can be observed both upstream and downstream of the reference shock position.

Additional knowledge can be brought with the use of the PDFs. It can be very useful to detect some rare events that can not be revealed from the standard deviation distributions. For the wall pressure, the PDFs distribution are almost centered around the most probable value which also corresponds to the mean stochastic *Kp* value for $x/c < 0.35$. This is no more the case in the area where the flow response is non-linear. The PDFs exhibit a wider range of *Kp* values with two dominant peaks. As a consequence, in this region, the mean stochastic *Kp* differs from the most probable *Kp* magnitude. For $x/c > 0.65$, the shock-wave is always located upstream independently of M_∞ in the uncertainty bounds considered and a narrow peak is observed on the PDFs. The *Kp* values are almost unsensitive to the uncertainty as previously observed on both the mean stochastic *Kp* values and σ_{Kp}. It is obvious that the skin-friction behaves quite differently because of the quite large variations observed in all the possible *Cf* values for $x/c > 0.35$. Once again, in this region of the airfoil, it is clear that the most probable *Cf* magnitude highly differs from the mean stochastic value.

This is in accordance with the results shown in Fig. 9 which depicts the spatial distribution of the Mach number coefficient of variation c_v. The coefficient of variation is a non-dimensional number and is a measure of dispersion of a probability distribution. It is defined as the ratio between the standard deviation σ and the mean stochastic value μ. Two distinct areas with high c_v magnitudes can be isolated. The first one, for $0.35 < x/c < 0.65$, corresponds to the region where non-linear variations of the pressure coefficient were underlined (Fig. 7) due to the variable shock position. In this region, dispersion as high as 30% can be observed. The second area of high variability is located in the boundary-layer behind the reference shock position. On the lower figure, a coefficient superior to one can be observed very locally whereas all the boundary-layer area exhibits c_v values superior to 0.5. These high dispersion values agree well with the observations previously drawn dealing with the boundary-layer separation downstream the shock when $M_\infty > 0.73$.

3.3. Coupling process

A sensitivity analysis can be performed by using the Sobol' decomposition (Saltelli and Sobol' [20]). It allows to determine the relative influence of each stochastic parameter on the system within the uncertainty range investigated. Thanks to the methodology used in the present study, the polynomial chaos based Sobol' indices can be directly calculated from the expansion coefficients.

Using the Sobol' decomposition of the total variance, we can write:

$$\sigma^2_{Total} = \sigma^2_M + \sigma^2_\alpha + \sigma^2_{M-\alpha}, \tag{9}$$

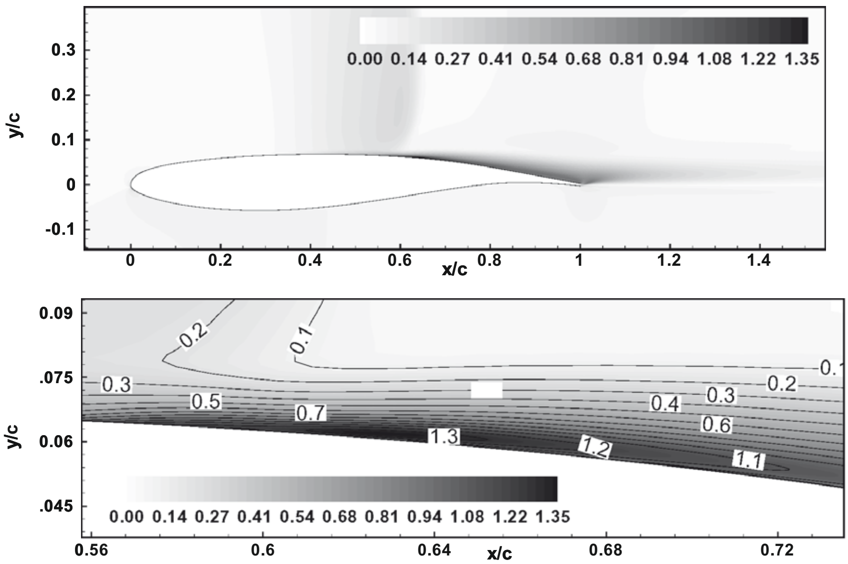

Fig. 9. Spatial distribution of the Mach number coefficient of variation c_v (top: whole airfoil, bottom: detailed view).

where σ_{Total} is the total standard deviation whereas σ_M and σ_α are the partial standard deviations respectively due to the Mach number and the angle of attack uncertainties. The $\sigma_{M-\alpha}$ term is the standard deviation resulting from the coupling process between the 2 stochastic parameters.

Fig. 10 presents the distribution of the partial standard deviations as well as the reference standard deviations from the mono-dimensional cases which have been noted, respectively $\sigma_M(Ref.)$ and $\sigma_\alpha(Ref.)$. The observations which can be drawn from both figures (a) and (b) are similar. σ_{Total} and σ_M exhibit the same shape and almost the same amplitudes meaning that σ_M dominates the whole standard deviation, meaning that the compressibility effect are the most influential. This result could have been guessed looking at the mono-dimensional cases, far less expensive in computing resources. The physical rationale for that is that variations in the Mach number induces a change of both the shock location and the shock strength on the suction-side, these two effects having a deep impact on the boundary-layer state downstream the shock. But there is also a significant coupling between compressibility effects and incidence effects resulting in a coupling term $\sigma_{M-\alpha}$ with superior magnitude to σ_α in the interaction region ($0.3 < x/c < 0.65$). Moreover, when both the mono- and bi-dimensional cases are compared, discrepancies appear both in magnitude and shape of the standard deviation. A 20% increase of the pick value is observed in the 2-parameter case compared to the mono-dimensional one for both the pressure and the skin-friction coefficients. One can also observe the absence of the bump in the standard deviation distributions $\sigma_M(Ref.)$ and $\sigma_\alpha(Ref.)$ due to the coupling process in the bi-dimensional case. This coupling process is much more evident on $\sigma_{M-\alpha}$ related to the skin-friction coefficient where the coupling term reaches a 40% value of the whole standard deviation in the interaction region. The conclusion is that non-linear stochastic interactions between the shock displacement and the suction-side boundary-layer downstream the shock are more important than isolated angle of attack effects.

4. Conclusions

The present work was aimed at investigating the sensitivity of a 2D transonic flow around an airfoil to uncertain parameters with the use of the Polynomial Chaos methodology. The stochastic inputs chosen in the present study are the infinite Mach number and the angle of attack due to the sensitivity of the flow to these variables in the transonic regime. The physical response of the flow to such uncertainties is studied based on bi-dimensional chaos simulations. The stochastic collocation methodology

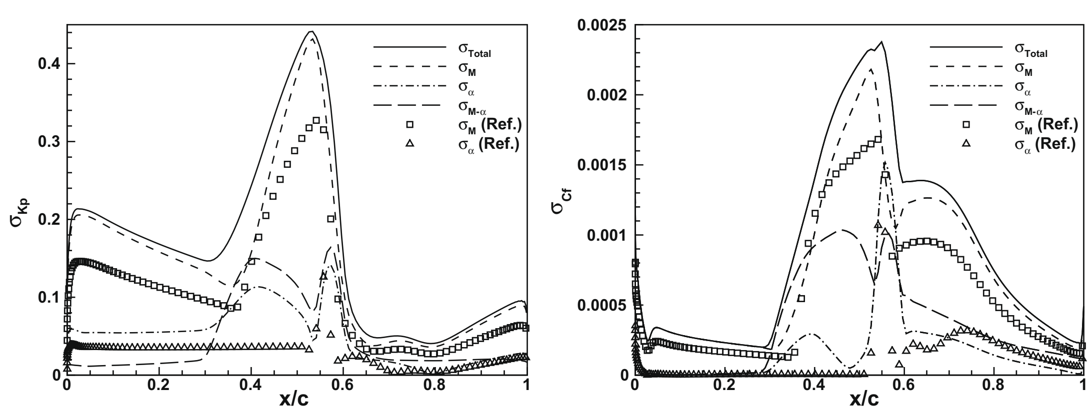

Fig. 10. Standard deviations derived from the Sobol' decomposition – Standard deviations extracted of the mono-dimensional simulations have been added (winward side only).

has succeeded in providing converged solutions up to the second-order. However, a spatial distinction can be made due to the different non-linearities involved in the flow. Thus, the pressure discontinuities (shocks) require the use of high-order polynomial expansions due to the stochastic parameters on the steep dependency of the shock position. On the contrary, the existence of non-linearities through the appearance of separation behind the mean shock position does not require high-order terms. Discrepancies appear between the most probable pressure and skin-friction distributions and the deterministic case in the interaction region where the shock moves dependently of the stochastic parameters which demonstrate the influence of the uncertainties on the response of the flow. The analysis of the PDFs distributions is also helpful to evidence the highly non-linear regions of the flow and investigate the rare events which can occur in these area. Another important observation is that prediction of the range of variation around the stochastic mean value and the standard deviation is not accurate in such a non-linear case: extreme events occur which are much stronger. Then, a detailed analysis of the coupling between the random parameters thanks to the Sobol' decomposition has been performed and revealed to be a powerful tool to analyse the sensitivity of the flow. The partial standard deviations differ from their mono-parameter counterparts both in shapes and magnitudes revealing that the study of the whole multi-parameter case is required in order to get accurate results. Another important conclusion drawn from Fig. 1 is that the most probable value (i.e. the one with the highest probability) of *Kp* and *Cf* at a given location may be very different from the mean stochastic value (found by integrating the pdf), because the pdf exhibit several significant peaks, each peak being associated to a pattern of the solution. Therefore, recovery of the full pdf profile appears to be mandatory for safety studies.

There is currently work in progress dealing with advanced methodologies aimed at reducing the cost of the quadrature evaluation for large multi-parameter stochastic space through the use of cubatures.

This work has been undertaken through a cooperation program between Université Pierre et Marie Curie (UPMC), the French national aerospace research center ONERA and Airbus Indutries.

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